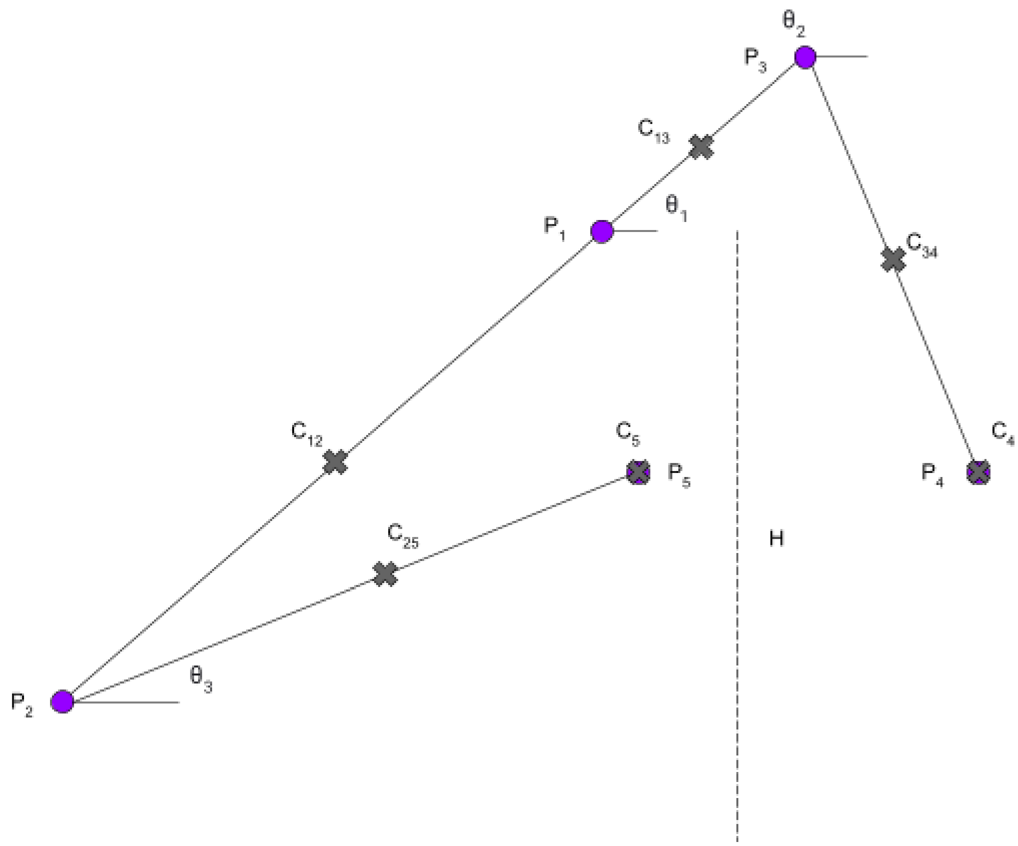
Whipper Trebuchet Dynamics



Givens	
m_{12},L_{12}	*mass and length of the long arm*
m_{13},L_{13}	*mass and length of the short arm*
m_{34},L_{34}	*mass and length of the counterweight arm*
m_{25},L_{25}	*mass and length of the sling*
m_4	*mass and length of the counterweight*
m_5	*mass and length of the projectile*
H	*height of the pivot*
θ_1	*angle between the long arm and the horizontal*
θ_2	*angle between the counterweight and the horizontal*
θ_3	*angle between the sling and the horizontal*
I_{12},I_{13},I_{34},I_{25}	*moments of inertia of the links (*$\frac{BH^3}{12}$ *for a rectangular bar)*
g	*gravitational acceleration*

Points and Vectors:

$P_1 = (0, H)$
$P_2 = P_1 + (L_{12}cos\theta_1, L_{12}sin\theta_1)$
$P_3 = P_1 - (L_{13}cos\theta_1, L_{13}sin\theta_1)$
$P_4 = P_3 + (L_{34}cos\theta_2, L_{34}sin\theta_2)$
$P_4 = P_2 + (L_{25}cos\theta_3, L_{25}sin\theta_3)$

$R_{12} = P_2 - P_1$
$R_{13} = P_3 - P_1$
$R_{34} = P_4 - P_3$
$R_{25} = P_5 - P_2$

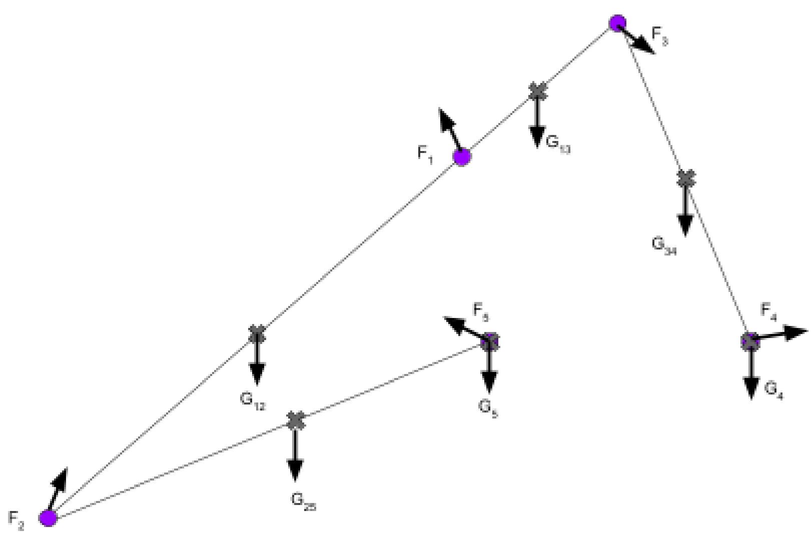
$C_{12} = P_1 + \frac{1}{2}R_{12}$
$C_{13} = P_1 + \frac{1}{2}R_{13}$
$C_{34} = P_3 + \frac{1}{2}R_{34}$
$C_{25} = P_2 + \frac{1}{2}R_{25}$
$C_4 = P_4$
$C_5 = P_5$

Accelerations

$$\ddot{P}_1 = (0, 0)$$
$$\ddot{P}_2 = (-\ddot{\theta}_1 L_{12} sin\theta_1 - \dot{\theta}_1^{\,2} L_{12} cos\theta_1,\ \ddot{\theta}_1 L_{12} cos\theta_1 - \dot{\theta}_1^{\,2} L_{12} sin\theta_1)$$
$$\ddot{P}_3 = (\ddot{\theta}_1 L_{13} sin\theta_1 + \dot{\theta}_1^{\,2} L_{13} cos\theta_1, -\ \ddot{\theta}_1 L_{13} cos\theta_1 + \dot{\theta}_1^{\,2} L_{13} sin\theta_1)$$
$$\ddot{P}_4 = \ddot{P}_3 +\ (\ddot{\theta}_2 L_{34} sin\theta_2 + \dot{\theta}_2^{\,2} L_{34} cos\theta_2, -\ \ddot{\theta}_2 L_{34} cos\theta_2 + \dot{\theta}_2^{\,2} L_{34} sin\theta_2)$$
$$\ddot{P}_5 = \ddot{P}_2 +\ (\ddot{\theta}_3 L_{25} sin\theta_3 + \dot{\theta}_3^{\,2} L_{25} cos\theta_3, -\ \ddot{\theta}_3 L_{25} cos\theta_3 + \dot{\theta}_3^{\,2} L_{25} sin\theta_3)$$

$$\ddot{C}_{12} = \frac{1}{2}\ddot{P}_2$$
$$\ddot{C}_{13} = \frac{1}{2}\ddot{P}_3$$
$$\ddot{C}_{34} = \ddot{P}_3 + \frac{1}{2}(\ddot{P}_4 - \ddot{P}_3)$$
$$\ddot{C}_{25} = \ddot{P}_2 + \frac{1}{2}(\ddot{P}_5 - \ddot{P}_2)$$
$$\ddot{C}_4 = \ddot{P}_4$$
$$\ddot{C}_5 = \ddot{P}_5$$

Free Body Diagram



Dynamics

$$G_{12} = (0, m_{12}g)$$
$$G_{13} = (0, m_{13}g)$$
$$G_{34} = (0, m_{34}g)$$
$$G_{25} = (0, m_{25}g)$$
$$G_4 = (0, m_4g)$$
$$G_5 = (0, m_5g)$$

$$I_{12}\ddot{\theta}_1 = -\frac{1}{2}R_{12} \times F_1 + \frac{1}{2}R_{12} \times F_2$$
$$I_{13}\ddot{\theta}_1 = -\frac{1}{2}R_{13} \times F_1 + \frac{1}{2}R_{13} \times F_3$$
$$I_{34}\ddot{\theta}_2 = -\frac{1}{2}R_{34} \times F_3 + \frac{1}{2}R_{34} \times F_4$$
$$I_{25}\ddot{\theta}_3 = -\frac{1}{2}R_{25} \times F_2 + \frac{1}{2}R_{25} \times F_5$$

$$m_{12}\ddot{C}_{12} = G_{12} + F_2 + F_1$$
$$m_{13}\ddot{C}_{13} = G_{13} + F_3 - F_1$$
$$m_{34}\ddot{C}_{34} = G_{34} - F_3 - F_4$$
$$m_{25}\ddot{C}_{25} = G_{25} - F_2 - F_5$$
$$m_4\ddot{C}_4 = G_5 + F_4$$
$$m_5\ddot{C}_5 = G_5 + F_5$$

Solving the nonlinear system of equations: implemented in python with both Scipy hybrid method and pymunk physics engine.

Material Choice

1) *Shaft Diameter*
   Solving for a 1566 Steel Cylindrical Shaft, Simply Supported
   Max Deflection: $\frac{PL^3}{48EI}$
   Max Bending Stress: $\frac{PL}{4I}$
   I =$\frac{\pi r^4}{4}$
   E = $30 * 10^6$psi
   L = 7in
   P = 50lbs

   Minimum diameter to keep deflection under .001in: .246in
   Minimum diameter to keep stress under yield strength (149ksi): .055in
   Minimum common size: .375 in.

2) *Arm Material* + *Size*
   Solving for a 3ft arm, Cantilever Beam
   Max Deflection: $\frac{PL^3}{3EI}$
   Max Stress: PL
   I = $\frac{BH^3}{12}$
   E = $1.8 * 10^6$psi
   L = 36in
   P = $ma = 12lbs * 92ft/s^2 = 120.4$ lbs (obtained via simulation)

   Max deflection with 1x2 Pine lumber: 1.56in
   Max stress with 1x2 Pine lumber:4334 psi
   This should work, yield strength is around 6000 psi
   1.56inch deflection is not ideal.

Predictions

I estimate that with a 30lb weight, 1-0.2-0.8-0.9 long,short,counterweight,sling ratio and a tennis ball projectile, this design can reach >95 mph launch velocity and >200 ft range with no wind.